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ON L-FUNCTIONS AND INTERTWINING OPERATORS FOR UNITARY GROUPS

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ABSTRACT

The ingredients of an "L-function machine" for the quasi-split group $U_{n,n+1} \times \operatorname{Res} \mathrm{GL}_n$ are treated here, following similar theories of P. Shapiro and S. Gelbart. We start with a known Rankin-Selberg type integral having an Euler product. In section 2 we compute the local integral to get a local L function. This is done by working with an "L group" related to ${}^L G$ and the relative root system. All computations are carried out for the split and the non-split case. In section 3 we address the problem of analytic continuation of the Eisenstein series. This involves computation of poles of intertwining operators.

1. Preliminaries

(1.1) Introduction

In this paper, we make some contributions to the theory of automorphic L-functions for the quasi-split unitary group in $2n+1$ variables $U_{n,n+1}$, using a Rankin–Selberg type integral representation. A similar theory was introduced for $U_{2,1}$ in [Ge. P.S. 1], and in [Ge. P.S. 2] a similar discussion is to be found for split groups of type $B_n \times A_{n-1}$. In all these cases, it is assumed that the automorphic representations in question are generic.

Since the group $\operatorname{Res}_F^E \mathrm{GL}_n$ sits inside $U_{n,n+1}$ as the Levi component of a maximal parabolic subgroup P of the subgroup $U_{n,n}$, it is natural to apply the "L-function machine" described in [Ge. Sh.] to the group $G' = G \times \operatorname{Res}_F^E \mathrm{GL}_n$, where $G = U_{n,n+1}$. If F denotes a global field, and E the quadratic extension defining G, we can consider a cuspidal generic representation π (resp. τ) of $G_{\mathbf{A}_F}$ (resp. $\operatorname{Res}_F^E(\mathrm{GL}_n)_{\mathbf{A}_F}$).

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To each holomorphic finite-dimensional representation r of ${}^LG$ (the Langlands dual group of G) one may attach an automorphic L-function

$$L(s,\pi\times\tau,r)=\prod_{v\ \text{unramified}} L(s,\pi_v\times\tau_v,r)$$

where $L(s,\pi_v\times\tau_v,r)$ is the "local Langlands factor" ([La 1]). By the well-known conjecture of Langlands, this Euler product should continue to a meromorphic function in the whole plane, with finitely many poles, also satisfying a functional equation relating s to $1-s$, called "the global functional equation".

To prove this we use the "L-function machine" alluded to above. At first we introduce a global zeta integral interpolating the automorphic L-function $L(s,\pi\times\tau,r)$, that is

$$I(s)=\int_{(U_{n,n})_F\backslash(U_{n,n})_{\mathbf{A}_F}} f_\pi(h)E^N_{f^\tau}(h,s)\,dh$$

where

$$E^N_{f^\tau}(h,s)=d_\tau(s)\sum_{\gamma\in P_F\backslash H_F} f^\tau(\gamma h)$$

(see also below). Here $E^N_{f^\tau}(h,s)$ is the so-called normalized Eisenstein series on $H=U_{n,n}$ attached to the function f^τ in the induced space $\mathrm{Ind}_P^H\,\tau|\det|^3$, and f_π is a cusp form on $U_{n,n+1}$ in the space of π. It is well known that $E^N_{f^\tau}(h,s)$ converges absolutely and uniformly in some right half plane. Now since the restriction of f_π to $H_{\mathbf{A}_F}$ is rapidly decreasing, the analytic properties of the zeta integral are determined by those of $E^N_{f^\tau}(h,s)$.

In the author's thesis [Ta], it is shown that $I(s)$ factors as an (Euler) product of local zeta integrals of the form

$$\zeta(W,f^\tau,s)=\int_{U_F^H\backslash H_F} W(h)W_{f^\tau}(h,s)\,dh$$

where F now denotes a local field, U_F^H the maximal unipotent subgroup of H_F, $W(h)$ a function in the Whittaker model of the local component of π, and $W_{f^\tau}(h,s)$ a kind of Whittaker function for f^τ. This is Step 1 of the "L-function machine".

As we shall see below, this integral can be explicitly computed in the unramified case, and shown to equal a rational function in q_F^{-s} (q_F the order of the residue field). In general, such an integral converges in some right half plane and satisfies a functional equation called "the local functional equation", which constitutes Step 3 of the "L-function machine" (see [Ta]).

In Section 2, we shall explicitly compute the integrals $\zeta(W,f^\tau,s)$ for unramified data. In case E remains non-split over F, we need first to interpret the well-known formula of [Cass. Sh.] for class 1 Whittaker functions in terms of character formulas for finite-dimensional representation of an L group closely related to ${}^LG$ and the relative root system (Proposition 1); the computation of the unramified zeta integral then involves classical character identities from [We]. On the other hand, if E splits over F, then $G(F) = \mathrm{GL}_{2n+1}(F)$, and the unramified zeta integral is computed via an expansion of Schur functions for GL_{2n+1} in terms of Schur functions for GL_n. In both cases, we obtain the result (Theorem 1):

$$(*) \qquad \zeta(W,f^\tau,s) = L(s,\pi\times\tau,r)$$

where $L(s,\pi\times\tau,r)$ is the Langlands L-function attached to the representation r of ${}^LG'$ induced from the representation $\rho_{2n+1}\otimes\rho_n\otimes 1$ of the connected component $({}^LG')^0 = \mathrm{GL}_{2n+1}(C)\times\mathrm{GL}_n(C)\times\mathrm{GL}_n(C)$. Implicit in (*) is the presence of a local normalizing factor $d_\tau(s)$ (a certain L-function which is attached to the group $\mathrm{Res}_F^E\,\mathrm{GL}_n$, and comes from the normalization of the global Eisenstein series). This completes Step 4 of the "L-function machine".

In order to address Step 2, it remains "only" to treat the meromorphic behaviour of the global zeta integral, i.e., the Eisenstein series. This is the subject matter of Section 3 of this paper.

By the theory of Langlands (cf. [La]), it suffices to analyze one intertwining operator $A(s,\tau,w)$, where the Weyl group element w "preserves" the maximal parabolic $P = (\mathrm{Res}_F^E\,\mathrm{GL}_n)\times U^P$. In case the inducing representation τ is not cuspidal, we first write down a reduction to the case of intertwining operators for maximal parabolic subgroups and cuspidal representations (Proposition 2). In principle, this reduction is already carried out by Theorem 2.1.1 of [Sh 2]; however, we need to carry it out in our specific case in order to write down explicitly which intertwining operators arise. For these "cuspidal induced from maximal parabolic" intertwining operators, we then show (following ideas of [Olsh]) that for a certain polynomial P in $C[\bar{q}_F^s, q_F^s]$,

$$P \qquad A(s,\tau,w)f^\tau$$

is always entire. (A similar polynomial, but of a higher degree, exists already from the general theory of Harich-Chandra; cf. [Sh 2], Section 2.)

(1.2) Notation

(a) Analysis of G as a group over F

Let F be a local non-archimedean field, E a quadratic extension of F. Write each element in E as $a + ib$ where a, b is in F, $E = F(i)$. Let O_F (resp. P_F) denote the ring of integers (resp. the prime ideal) of F, ω a uniformizer for F, and q_F the order of the residue field.

Let V be a $2n+1$ dimensional vector space over E equipped with a bilinear Hermitian form $(\ , \ )$. Suppose that V has a maximal isotropic subspace of dimension n and that a matrix form of $(\ , \ )$ is

$$J = \underbrace{\begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}}_{2n+1}.$$

Let $G = U_{n,n+1}$, the group of isometries for $(\ , \ )$, i.e. $G = \{g \in \mathrm{GL}_{2n+1}(E) : \bar{g}^t J g = J\}$. One may regard G as the F rational points of a quasi-split algebraic group $\tilde{G}$ which is defined over F, split over E, and whose group of E rational points is $\mathrm{GL}_{2n+1}(E)$. To do so, we look at V as a $4n+2$ dimensional vector space over F. If $e_1, \ldots, e_{2n+1}$ is the basis in which we wrote the matrix form, then $e_1, \ldots, e_{2n+1}, ie_1, \ldots, ie_{2n+1}$ may serve as a basis for V over F. Clearly, in this basis each element of G corresponds to an element g of $\mathrm{GL}_{4n+2}(F)$ of the form

$$g = \left[\begin{array}{c|c} A & i^2 B \\ \hline B & A \end{array}\right]$$

satisfying the additional polynomial condition

$$\left[\begin{array}{c|c} A^t & -i^2 B^t \\ \hline -B^t & A^t \end{array}\right] \left[\begin{array}{c|c} J & 0 \\ \hline 0 & J \end{array}\right] \left[\begin{array}{c|c} A & i^2 B \\ \hline B & A \end{array}\right] = \left[\begin{array}{c|c} J & 0 \\ \hline 0 & J \end{array}\right].$$

(b) Description of key subgroups H and P

Denote the maximal isotropic subspace of V (as a vector space over E) by X, then write

$$V = X + (l) + X^{\vee}.$$

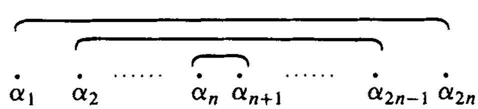
where $X^\vee$ is an isotropic subspace of V dual to X, and (l) is the 1-dimensional anisotropic space orthogonal to $X + X^\vee$. Let H denote the subgroup of G fixing (l), so that $H = U_{n,n}$, the isometry group of $X + X^\vee$. Let P be the stabilizer of X in H; it is a maximal parabolic subgroup of H, and one denotes its unipotent radical by U^P. For convenience, pick a basis $x_1, \ldots, x_n, l, x_1^\vee, \ldots, x_n^\vee$ for V, so that the corresponding matrix forms of H and P look like

$$\begin{bmatrix} & & & 0 & & & \\ & * & & \vdots & & * & \\ & & & 0 & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & 0 & & & \\ & * & & \vdots & & * & \\ & & & 0 & & & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} & & & 0 & & & \\ & * & & \vdots & & * & \\ & & & 0 & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & 0 & & & \\ & \mathbf{0} & & \vdots & & * & \\ & & & 0 & & & \end{bmatrix}.$$

(c) Root systems and non-degenerate characters of U^G

By G, T, T_d, we mean the appropriate linear algebraic group defined over F. Let T be the maximal torus of G. Let T_d be the maximal split torus of G. Put T_d in T. The root system of G with respect to T is of type A_{2n}. This is the root system we get by the nontrivial weights of the adjoint action of T on n, the Lie algebra of U^G. Denote this root system by Φ. Now restrict the adjoint action to T_d to get a new root system which is of type BC_n. Denote this root system by Σ.

There exists an action of the Galois group $\mathrm{Gal}(E:F)$ on Φ, preserving the simple roots and such that each orbit in Φ corresponds to a root in Σ. Also, orbits of simple roots of Φ go to simple roots of Σ. This correspondence is obtained by restricting a root in Φ to T_d. Denote the non-trivial action of the Galois group by γ. This action on the Dynkin diagram look like this:



Define a non-degenerate character in the same form as in [Cass. Sh.]. Then following [Sh 1], section 3, one can introduce a global non-degenerate character which locally agrees with that character ([Ta]).

(d) *L* groups

The L group of $U_{n,n+1}$

For any torus A in G let $X(A)$ (resp. $Y(A)$) be the group of characters of A (resp. the group of one parameter subgroups of A). Let also ${}^LA$ be the complex (split) torus such that $X({}^LA) = Y(A)$.

The L group of $U_{n,n+1}$ is $\mathrm{GL}_{2n+1}(C) \rtimes \Gamma$ where Γ is $\mathrm{Gal}(E{:}F)$. The action of Γ on the F rational points of the maximal torus T produces an action on $X(T)$. Identify $X(T)$ with Z^{2n+1}; then this action takes the form:

$$(n_1, \dots, n_{2n+1}) \to (-n_{2n+1}, \dots, -n_1).$$

This action passes roots to roots and simple roots to simple roots. In turn, Γ acts on $Y(T)$ or on $X({}^LT)$, preserving the root system. The corresponding induced action of Γ on $\mathrm{GL}_{2n+1}(C)$ is as follows:

$$\gamma : g \to j^{-1} g^t j$$

where

$$j = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & -1 & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix} \Bigg\} 2n+1 .$$

For each local class 1 representation of $U_{n,n+1}$, we recall now the definition of the corresponding semi-simple conjugacy class in ${}^LG$.

The inclusion of $Y(T_d)$ in $Y(T)$ or of $X({}^LT_d)$ in $X({}^LT)$ as the fixed point set of γ induces a surjective homomorphism $\nu : {}^LT \to {}^LT_d$. We can choose ${}^LT_d = \{(x_1, \dots, x_n, 1, x_n^{-1}, \dots, x_1^{-1});\ x_i \in C^*\}$ and hence $\nu(t) = t \cdot t^\gamma$. Note that ${}^LT_d$ is determined only up to its dimension, and indeed in the next chapters we shall identify it with the maximal torus of $\mathrm{SP}_n(C)$.

Let A be the endomorphism of ${}^LT$ defined by $t \to t^{-1} \cdot t^\gamma$. Put $U = (\operatorname{Ker} A)^0$, $V = \operatorname{Im} g(A)$, then

$$U = \{(x_1, \dots, x_n, 1, x_n^{-1}, \dots, x_1^{-1});\ x_i \in C^*\}, \qquad \text{and}$$

$$V = \{(x_1, \dots, x_n, y, x_n, \dots, x_1);\ y, x_i \in C^*\}.$$

Also $\nu(V) = 1$, $\nu(U) = {}^LT_d$, and ${}^LT = U \cdot V$ (cf. [Bo], p. 37).

Now suppose π is parametrized by the unramified character $\chi = \chi(\pi)$. Then by the Satake homomorhism (cf. [Cart], p. 146 (19), p. 149 (28)) there corresponds an element t_d^χ of the complex torus ${}^LT_d$ given by the formula

$$\chi(t) = v(t)(t_d^\chi) \qquad \text{for all } t \text{ in } Y(T_d) \quad \text{(see also 2.1)} \tag{1.2.1}$$

where v is the isomorphism: $Y(T_d) \to X({}^LT_d)$ (v for "valuation"). It is easy to see that

$$t_d^\chi = (t_1^\pi, \ldots, t_n^\pi, t_n^{\pi^{-1}}, \ldots, t_1^{\pi^{-1}}) \quad \text{where } t_i^\pi = \chi(1, \ldots, \underset{\substack{\uparrow \\ i\text{th place}}}{\omega}, \ldots, 1, \ldots, \omega^{-1}, \ldots, 1).$$

So each class 1 representation goes to an orbit of W in ${}^LT_d$ where $W = N({}^LT_d)/{}^LT_d$. According to [Bo] (Lemma 6.4 and Lemma 6.5), there exist bijective maps $\bar{\mu}, \bar{\nu}$ such that

$${}^LT_d/W \xrightarrow{\bar{\nu}^{-1}} {}^LT \rtimes \gamma/\mathrm{Int}\, N({}^LT_d) \xrightarrow{\bar{\mu}} ({}^LG^0 \rtimes \gamma)_{ss}/\mathrm{Int}\, {}^LG^0$$

with $\bar{\nu}$ induced by $\nu' : {}^LT \rtimes \gamma \to {}^LT_d$ where $\nu'(t,\gamma) = \nu(t)$. Then there exists an $\mathrm{Int}\, N({}^LT_d)$ orbit in ${}^LT \rtimes \gamma$ which corresponds to the orbit of t_d^χ under W. Take the $\mathrm{Int}\, N({}^LT_d)$ orbit of (t^χ, γ) where

$$t^\chi = (t_1^\pi, \ldots, t_n^\pi, 1, \ldots, 1).$$

Note that $\nu'(t^\chi, \gamma) = t_d^\chi$.

In the split case, the representation π is a representation of $\mathrm{GL}_{2n+1}(E)$ and the semi-simple conjugacy class has an easily computed representative in ${}^LT \rtimes \Gamma$ in the form of $(t^\chi, 1)$, where $t^\chi = (t_1^\pi, \ldots, t_{2n+1}^\pi)$.

The group $\mathrm{Res}_F^E\, \mathrm{GL}_n$

The L group is $\mathrm{GL}_n(C) \times \mathrm{GL}_n(C) \rtimes \Gamma$ where the action of Γ on $\mathrm{GL}_n(C) \times \mathrm{GL}_n(C)$ takes (a,b) to (b,a). Given a class 1 representation π parametrized by the character χ of T, we define the class t^χ in the L group as follows. In the split case, $\mathrm{Res}_F^E\, \mathrm{GL}_n(F) = \mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$, and χ corresponds to a pair of characters (χ', χ''). So put $t^\chi = (t^{\chi'}, t^{\chi''}, 1)$ where $t^{\chi'}$ (resp. $t^{\chi''}$) is the class in $\mathrm{GL}_n(C)$ corresponding to the class 1 representation $\pi(\chi')$ (resp. $\pi(\chi'')$). In the non-split case put $t^\chi = (t^\chi, 1, \gamma)$ where t^χ is the class in $\mathrm{GL}_n(C)$ corresponding to the class 1 representation $\pi(\chi)$.

2. Calculation of Unramified Zeta Integrals

(2.1) A character type formula for the class 1 Whittaker function of G

We assume that G is the completion of the quasi-split unitary group $U_{n,n+1}$ at a prime of the global field. Thus G is either the F rational points of a quasi-split unitary group defined over F (F is local), or the F rational points of GL_{2n+1}. We assume also that G is unramified, i.e. in the non-split case E is unramified over F.

Now suppose π is any class 1 representation of G. Realizing the representation π in a space of Whittaker functions, it is well known that there exists a unique function $\mathbf{W}_{\pi(\chi)}$ in this model which is invariant under the standard maximal compact subgroup K, and equal to 1 at the identity. The purpose of this section is to give a formula for this function in terms of a character of a representation of some classical group, and to compute the resulting unramified zeta integrals (involving $\mathbf{W}_{\pi(\chi)}$ and already referred to briefly in Section 1).

Recall T is a maximal torus of G, T_d a maximal split torus contained in T, $X(T_d)$ (resp. $Y(T_d)$) the character group of T_d (resp. the group of one parameter subgroups of T_d). Note $T_d = (F^*)^n$ and $X(T_d) = Z^n$. Let $\langle\ \rangle$ be the natural pairing between $X(T_d)$ and $Y(T_d)$ defined by

$$t^{\langle y,\lambda\rangle} = \lambda(y(t))$$

where λ is in $X(T_d)$, y in $Y(T_d)$, and t in G_m, the multiplicative group of F. There exists an onto homomorphism $\mathrm{ord}_{T_d}\colon T_d \to Y(T_d)$ defined by

$$\langle \mathrm{ord}_{T_d}(h),\lambda\rangle = v(\lambda(h)) \qquad \text{for all } \lambda \text{ in } X(T_d),$$

where v is the valuation in F. Then $Y(T_d) = T_d(F)/T_d^0(F)$ where $T_d^0(F) = T_d(F) \cap K$. Recall ${}^LT_d$ is a torus over C s.t. $Y(T_d) \overset{v}{=} X({}^LT_d)$. Let $C[Y(T_d)]$ be the group algebra of the free abelian group $Y(T_d)$. Then $C[Y(T_d)] = C[X({}^LT_d)]$ and, since ${}^LT_d$ is split, $C[X({}^LT_d)] = C[({}^LT_d)]$, the polynomial algebra of ${}^LT_d$.

Let $\pi = \mathrm{ind}^G_{T(F)\cup G(F)}\chi$ where χ is an unramified character of $T(F)$. By [Bo], p. 43, $T(F)/T^0_{(F)} = T_d(F)/T^0_d(F)$ so χ can be regarded as a homomorphism of $Y(T_d)$. Extend χ to $C[Y(T_d)]$ or to $C[({}^LT_d)]$. Then, as a homomorphism of a polynomial algebra, it is defined by an evaluation, i.e. there exists t_d^χ in ${}^LT_d$ such that $\chi(t) = v(t)(t_d^\chi)$ for all t in $Y(T_d)$ (see also 1.2(d)).

Our purpose now is to explain the Whittaker function formula

$$\mathbf{W}_{\pi(\chi)}(t) = \sum_{w\in W_0} \prod_{\substack{\alpha>0\\ \alpha\in\Sigma^{nd}}} \frac{(w\chi\delta_G^{1/2})(t)}{1 - w\chi(a_\alpha^{-d_\alpha})},$$

which is proved in [Cass. Sh.]. In case $G = GL_{2n+1}$, this is the known formula of Shintani, expressing $\mathbf{W}_{\pi(\chi)}$ as a character formula for $GL_{2n+1}(C)$. To obtain a similar interpretation for $U_{n,n+1}$, we need to consider several root systems (which all coincide for split groups).

Recall Σ denotes the root system of G with respect to the maximal split torus. For any root system Φ, $\hat{\Phi}$ denotes the dual root system of Φ and Φ^{nd} the non-divisible roots of Φ, i.e.

$$\{\alpha \in \Phi;\ \alpha/2 \notin \Phi\}.$$

The following theorem is due to Bruhat and Tits and it is taken from [Mac 2] (Theorem 2.5.2, p. 28).

THEOREM. *Let G be a simply connected simple linear algebraic group. Then there exists a reduced irreducible root system Σ_0, subgroups N and U_α of G, where $\alpha \in \Sigma_{aff}$ (the affine root system of Σ_0), and a surjective homomorphism $h: N \to W$ (where W is the affine Weil group) such that the triple $(N, h, (U_\alpha)_{\alpha \in \Sigma_{aff}})$ is an affine root structure in the sense of* [Mac 2], Def. 2.5.3.

The root system Σ_0 is not, in general, the root system of G with respect to the maximal split torus. Let W_0 be the Weyl group of Σ_0. Define now a set Σ_1 as follows. For α in Σ_{aff} let q_α be the index $[U_{\alpha-1} : U_\alpha]$; then $\Sigma_0 \subset \Sigma_1 \subset \Sigma_0 \cup \frac{1}{2}\Sigma_0$, and for α in Σ_0, $\alpha/2$ lies in Σ_1 if and only if $q_{\alpha+1} \neq q_\alpha$ (cf. [Cass.], p. 390). Then Σ_1 is a root system with the same Weyl group W_0 (cf. [Mac 2], p. 38). Let $q_{\alpha/2} = q_{\alpha+1}/q_\alpha$ and define

$$d_\alpha = \begin{cases} 1 & \text{if } q_\alpha = q_{\alpha+1}, \\ 2 & \text{otherwise.} \end{cases}$$

Let V^* be the vector space in which Σ_0 is a root system. Let $\langle\ \rangle$ be the scalar product in V^*. For a in V^* let $\check{a}$ in V be the image of $2a/\langle a, a\rangle$ under the identification of V and V^*. The set $\{\check{\alpha}; \alpha \in \Sigma_0\}$ is a root system in V of dual type and denoted $\hat{\Sigma}_0$.

Identify $\check{\alpha}$ (for $\alpha \in \Sigma_0$) with the translation of V of the form $x \to x - \check{\alpha}$. Then $\check{\alpha}$ can be regarded as an element of W of the form $a_\alpha = w_\alpha \circ w_{\alpha-1}$. By the homomorphism $h: N \to W$, a_α is also a coset of T_d/T_d^0, since translations in W are the image of T/T^0 (see also [Mac 2], p. 16). By the relation 11 in [Cass.], p. 390 it is clear that the action of W_0 on a_α in T_d/T_d^0 agrees with the action of W_0 on $\check{\alpha}$ as a translation. Then a_α for $\alpha \in \Sigma_0$ can be looked at as the root system $\hat{\Sigma}_0$ lying in $T_d/T_d^0 \otimes R$. Therefore, Σ_0 can also be put inside $X(T_d) \otimes R$, the vector space in which Σ lies. Then each $\alpha \in \Sigma$ is a positive multiple of a unique root $\lambda(\alpha)$ in Σ_0 and the map λ is a bijection of Σ^{nd} and Σ_0 ([Cass.], p. 389).

LEMMA 1. *The set* $\{a_\alpha^{d_\alpha}\}_{\alpha\in\Sigma_0}$ *in* $T_d/T_d^0 \otimes R$ *is the root system* $\hat{\Sigma}_1^{nd}$.

PROOF. If $\alpha \in \Sigma_0$ and $\alpha/2 \notin \Sigma_1$ then $d_\alpha = 1$, $\alpha \in \Sigma_1^{nd}$, and $a_\alpha^{d_\alpha} = a_\alpha = \check{\alpha}$. If $\alpha \in \Sigma_0$ and $\alpha/2 \in \Sigma_1$ then $d_\alpha = 2$, α and $\alpha/2$ both lie in Σ_1, $\alpha/2 \in \Sigma_1^{nd}$, and $\alpha/2(a_\alpha^{d_\alpha}) = \alpha(a_\alpha) = 2$.

REMARK. For a reductive group (as opposed to a simply connected simple algebraic group), we take Σ_0 to be the root system of Bruhat and Tits with respect to the simply connected covering group of the derived group of G ([Cass.], p. 387).

LEMMA 2. *For* $U_{n,n+1}$, Σ_0 *is of type* C_n *and* Σ_1 *is of type* BC_n.

PROOF. It is enough to show that for some $\alpha \in \Sigma_0$, $q_{\alpha/2} \neq 1$, since this implies α and $\alpha/2$ both lie in Σ_1, and therefore Σ_1 is not reduced. The only nonreduced irreducible root system of degree n is of type BC_n, and therefore Σ_0 by its relation to Σ_1 must be of type C_n. To show that Σ is not reduced, take a *simple* root α in Σ (therefore in Σ^{nd}), let P_α be the parabolic of G corresponding to α, and M_α its Levi component. Let $\bar{G}_\alpha$ be the simply connected covering group of the derived group of M_α. There exists a simple root α such that $\bar{G}_\alpha = \mathrm{SU}_3$. Then by [Cass. Sh.] (p. 218), $q_{\lambda(\alpha)}$ and $q_{\lambda(\alpha)+1}$ are easily computed and are different. (Notice that in [Cass. Sh.], the notation q_α is used instead of $q_{\lambda(\alpha)}$; see also [Cass.], p. 394.)

We conclude that for $U_{n,n+1}$, Σ_1^{nd} and Σ^{nd} are of type B_n, and $\hat{\Sigma}_1^{nd}$ is of type C_n. From p. 226 of [Cass. Sh.] we have the following:

$$\mathbf{W}_\chi(t) = \sum_{w\in W_0} \prod_{\substack{\alpha>0 \\ \alpha\in\Sigma^{nd}}} \frac{(w\chi\delta_G^{1/2})(t)}{1 - w\chi(a_\alpha^{-d_\alpha})},$$

where $w\chi$ represents the image of χ under w. From the previous discussion in particular, Lemma 1 and 1.2.1 we can conclude the following:

PROPOSITION 1. *Let* $\overline{{}^L\mathbf{G}}$ *be the reductive split group over* C *with maximal torus* ${}^LT_d$ *and root system* $\hat{\Sigma}_1^{nd}$. *Then*

$$\mathbf{W}_\chi(t) = \mathbf{X}_{v(t)}^{\overline{{}^L\mathbf{G}}}(t_d^\chi)\delta_G^{1/2}(t)$$

where t *is in* $T_d^0\backslash T_d$ *regarded also as an element of* $V(T_d)$, $\underline{v(t)}$ *is the character of* ${}^LT_d$ *corresponding to* $\underline{t}$ *by the isomorphism* v, (1.2 d) $\mathbf{X}_{v(t)}^{\overline{{}^L\mathbf{G}}}$ *is the character of the representation of* $\overline{{}^L\mathbf{G}}$ *of highest weight* $v(t)$, *and* δ_G *is the modulus function of the Borel subgroup of* G. *One takes both sides of the formula to be* 0 *if* $v(t)$ *is not a highest weight.*

(2.2) Unramified computations: The non-split case

Our purpose is to compute the unramified zeta integral $\zeta(W,f^\tau,s)$ which comes to us via the global theory of Rankin–Selberg integrals described in the Introduction. In fact, the integral in question reduces to one of the form

$$\zeta(W,f^\tau,s) = \int_{T_d^0\backslash T_d} \mathbf{W}_\chi(t)\mathbf{W}_\tau^0(t)\delta_P^{1/2}(t)|t|^s\delta_H^{-1}(t)\,dt \tag{2.2.1}$$

where $\mathbf{W}_\tau^0(t)$ is the class 1 Whittaker function for the spherical representation τ of the group $\mathrm{GL}_n(E)$, δ_P (resp. δ_H) is the modulus function of P (resp. the Borel subgroup of H). We note that the factor $\delta_P^{1/2}(t)|t|^s$ arises from the definition of the (unitarily) induced space $\mathrm{Ind}_P^H\tau|\ \ |^s$, and the factor δ_H^{-1} arises from the integration formula corresponding to the Iwasawa decomposition $U^H T_d^0\backslash T_d K$. Note also that T_d is simultaneously regarded as the maximal split torus of G,H, and the copy of $\mathrm{Res}_F^E\,\mathrm{GL}_n$ in P.

By Proposition 1,

$$W_\chi(t) = \mathbf{X}^{SP_n}_{(n_1,\ldots,n_n)}(t_1^\pi,\ldots,t_n^\pi)\delta_G^{1/2}(t)$$

where $(t_1^\pi,\ldots,t_n^\pi,t_n^{\pi^{-1}},\ldots,t_1^{\pi^{-1}}) = t_d^\chi$ and $(n_1,\ldots,n_n) = v(t)$ (see also 1.2 (d)). On the other hand, by the well-known formula of [Shin],

$$W_\tau^0(t) = \mathbf{X}^{\mathrm{GL}_n}_{(n_1,\ldots,n_n)}(t_1^\tau,\ldots,t_n^\tau)\delta^{1/2}_{\mathrm{GL}_nE}(t)$$

where $(t_1^\tau,\ldots,t_n^\tau)$ is the class in $\mathrm{GL}_n(C)$ corresponding to τ. Also note that

$$\delta_G^{1/2}(t) = \delta^{1/2}_{\mathrm{GL}_nE}(t)\delta_P^{1/2}(t)|\det(t)|_E^{1/2} \quad\text{and}\quad \delta_H(t) = \delta_{\mathrm{GL}_nE}(t)\delta_P(t).$$

If in the definition of induced representation $\mathrm{Ind}\,\tau|\ \ |^s$ we replace s by $s-\frac{1}{2}$, then the local integral reads

$$\sum_{\substack{\text{all partitions } \lambda\\ \text{of length } n}} \mathbf{X}_\lambda^{\mathrm{SP}_n}(t_1^\pi,\ldots,t_n^\pi)\mathbf{X}_\lambda^{\mathrm{GL}_n}(t_1^\tau,\ldots,t_n^\tau)q_E^{-|\lambda|s}$$

where $|\lambda|$ is the weight of λ (in the sense of [Mac 1], i.e. $|\lambda| = \sum_j n_j$).

In the notation of Macdonald ([Mac 1], p. 24, or [Shin]),

$$\mathbf{X}_\lambda^{\mathrm{GL}_n}(t_1^\tau,\ldots,t_n^\tau) = \frac{\mathfrak{a}_{\lambda+\delta}(t_1^\tau,\ldots,t_n^\tau)}{\mathfrak{a}_\delta(t_1^\tau,\ldots,t_n^\tau)}$$

where $\delta = (n-1,\ldots,0)$ and

$$\mathfrak{a}_\lambda(t_1^\tau,\ldots,t_n^\tau) = \begin{vmatrix} t_1^{\tau^{f_1}} & \cdots & t_n^{\tau^{f_1}} \\ \vdots & & \vdots \\ t_1^{\tau^{f_n}} & \cdots & t_n^{\tau^{f_n}} \end{vmatrix},$$

for all highest weights $\lambda = (f_1,\ldots,f_n)$. Since

$$\frac{\mathfrak{a}_{\lambda+\delta}(at_1^\tau,\ldots,at_n^\tau)}{\mathfrak{a}_\delta(at_1^\tau,\ldots,at_n^\tau)} = \frac{\mathfrak{a}_{\lambda+\delta}(t_1^\tau,\ldots,t_n^\tau)}{\mathfrak{a}_\delta(t_1^\tau,\ldots,t_n^\tau)}(a)^{\Sigma f_i}$$

we let $t_i^\tau = t_i^\tau q_E^{-s}$ and absorb $q_E^{-s\Sigma n_i}$ in $\mathbf{X}_\lambda^{\mathrm{GL}_n}(t_1^\tau,\ldots,t_n^\tau)$. Now let

$$\varphi_\pi(t_j^\tau) = \prod_{1\le i\le n}(1 - t_i^\pi t_j^\tau)\prod_{1\le i\le n}(1 - t_i^{\pi^{-1}} t_j^\tau).$$

Then following [We], p. 220 (last formula),

$$\frac{\mathfrak{a}_\delta(t_1^\tau,\ldots,t_n^\tau)\prod_{i<j}(1-t_i^\tau t_j^\tau)}{\prod_{1\le j\le n}\varphi_\pi(t_j^\tau)}$$

$$= \sum_{f_1>f_2>\cdots>f_n>0} |t_\tau^{f_1+n-1},\ldots,t_\tau^{f_n}|\mathbf{X}^{\mathrm{SP}_n}_{(f_1,\ldots,f_n)}(t_1^\pi,\ldots,t_n^\pi).$$

Dividing both sides by $\mathfrak{a}_\delta(t_1^\tau,\ldots,t_n^\tau)$ gives

$$\frac{\prod_{i<j}(1-t_i^\tau t_j^\tau)}{\prod_{1\le j\le n}\varphi_\pi(t_j^\tau)} = \sum_\lambda \mathbf{X}_\lambda^{\mathrm{GL}_n}(t_1^\tau,\ldots,t_n^\tau)\mathbf{X}_\lambda^{\mathrm{SP}_n}(t_1^\pi,\ldots,t_n^\pi).$$

So if we recall that we put t_i^τ in place of $t_i^\tau q_E^{-s}$, the last formula reads

$$\frac{\prod_{i<j}(1-t_i^\tau t_j^\tau q_E^{-2s})}{\prod_{1\le i\le n}\left(\prod_{1\le j\le n}((1-t_i^\pi t_j^\tau q_E^{-s})(1-t_i^{\pi^{-1}}t_j^\tau q_E^{-s}))\right)}$$

$$= \sum_\lambda \mathbf{X}_\lambda^{\mathrm{GL}_n}(t_1^\tau,\ldots,t_n^\tau)\mathbf{X}_\lambda^{\mathrm{SP}_n}(t_1^\pi,\ldots,t_n^\pi)q_E^{-|\lambda|s}.$$

(2.3) Unramified computations: The split case

Here $G = \mathrm{GL}_{2n+1}(F)$ and $(\mathrm{Res}\,\mathrm{GL}_n)(F)$ becomes $\mathrm{GL}_n(F)\times\mathrm{GL}_n(F)$. Recall $W_\chi(t) = \mathbf{X}_{v(t)}^{\mathrm{GL}_{2n+1}}(t_1^\pi,\ldots,t_{2n+1}^\pi)\delta^{1/2}_{\mathrm{GL}_{2n+1}}(t)$ where $t = (t_1,\ldots,t_n,1,t_{n+1},\ldots,t_{2n})$

and $((t_1^\pi,\dots,t_{2n+1}^\pi),1)$ is the semi-simple conjugacy class of ${}^LG$ corresponding to the class 1 representation π. Recall now the definition of $\operatorname{ind}_P^H \tau' \otimes \tau''$. If $V' \otimes V''$ is the space on which $\operatorname{Res}_F^E \mathrm{GL}_n$ acts, then for g in the copy of $\operatorname{Res}_F^E \mathrm{GL}_n(F)$ inside P,

$$g = \begin{array}{|c|c|}\hline a & \\ \hline & Jb^{t^{-1}}J \\ \hline\end{array}$$

and $g(v' \otimes v'') = \tau'(a)v' \otimes \tau''(b)v''$. Therefore

$$W_\tau(t) = \mathbf{X}^{\mathrm{GL}_n}_{v(t_1,\dots,t_n)}(u_1,\dots,u_n)\mathbf{X}^{\mathrm{GL}_n}_{v(t_{2n}^{-1},\dots,t_{n+1}^{-1})}(v_1,\dots,v_n)\delta^{1/2}_{\mathrm{GL}_n\times\mathrm{GL}_n}(t),$$

where $((u_1,\dots,u_n),(v_1,\dots,v_n),1)$ is the semi-simple conjugacy class of ${}^L(\operatorname{Res}_F^E \mathrm{GL}_n)$ corresponding to the representation τ, and $v(t_1,\dots,t_n)$ is the p-adic valuation of $(t_1,\dots,t_n)$.

A similar computation as in the non-split case shows that

$$\delta_H = \delta_{\mathrm{GL}_n\times\mathrm{GL}_n}(t)\,\delta_P(t),$$

$$\delta_G(t) = \delta_{\mathrm{GL}_n\times\mathrm{GL}_n}(t)\delta_P(t)|\det(t_1,\dots,t_n)|_{F_v}|\det(t_{2n}^{-1},\dots,t_{n+1}^{-1})|_{F_v},$$

and

$$|t|^{s'}_{F_v} = |\det(t_1,\dots,t_n)|^{s'}_{F_v}|\det(t_{2n}^{-1},\dots,t_{n+1}^{-1})|^{s'}_{F_v}.$$

So again, we change s to $s - \frac{1}{2}$ to get

$$\zeta(W,f^\tau,s)$$

$$= \sum_{a_1>a_2>\cdots>a_n>0>-b_n>\cdots>-b_1} \mathbf{X}^{\mathrm{GL}_{2n+1}}_{(a_1,\dots,a_n,0,-b_n,\dots,-b_1)}(t_1^\pi,\dots,t_{2n+1}^\pi) \tag{2.3.1}$$

$$\times\, \mathbf{X}^{\mathrm{GL}_n}_{(a_1,\dots,a_n)}(u_1,\dots,u_n)\mathbf{X}^{\mathrm{GL}_n}_{(b_1,\dots,b_n)}(v_1,\dots,v_n)q^{-(\Sigma_i a_i+b_i)s}.$$

To compute (2.3.1) we first put $v_i = v_i q_{F_v}^{-s}$ and $u_i = u_i q_{F_v}^{-s}$; then $q^{-(\Sigma a_i)s}$ (resp. $q^{-(\Sigma b_i)s}$) is absorbed in $\mathbf{X}^{\mathrm{GL}_n}_{(a_1,\dots,a_n)}(u_1,\dots,u_n)$ (resp. $\mathbf{X}^{\mathrm{GL}_n}_{(b_1,\dots,b_n)}(v_1,\dots,v_n)$). Recall

$$\mathbf{X}^{\mathrm{GL}_{2n+1}} = \frac{\mathfrak{a}_{\lambda+\delta}(t_1^\tau,\dots,t_n^\tau)}{\mathfrak{a}_\delta(t_1^\tau,\dots,t_n^\tau)} \qquad \text{([Mac 1], p. 24; [Shin]).}$$

Now express the numerator's determinant in terms of its $(n+1,n+1)$ minors m which sit in the upper $n+1 \times 2n+1$ rectangular submatrix. We then get

$$\mathbf{X}^{\mathrm{GL}_{2n+1}}(t_1^\pi,\dots,t_{2n+1}^\pi)$$

$$= \sum_m \operatorname{sgn}(m) \frac{1}{\prod\limits_{\substack{i<j \\ \text{not both in same} \\ \text{subset of } t\text{'s}}} t_i - t_j} \mathbf{X}^{\mathrm{GL}_{n+1}}_{(a_1+n,\dots,a_n+n,n)}(t_{m_1},\dots,t_{m_{n+1}})$$

$$\times \mathbf{X}^{\mathrm{GL}_n}_{(-b_n,\dots,-b_1)}(t_{m_{n+2}},\dots,t_{m_{2n+1}})$$

where $(t_{m_1},\dots,t_{m_{n+1}})$ is the choice of $n+1$ elements from the set $\{t_1^\pi,\dots,t_{2n+1}^\pi\}$, defining the minor m. Notice

$$\mathbf{X}^{\mathrm{GL}_{n+1}}_{(a_1+n,\dots,a_n+n,n)}(t_{m_1},\dots,t_{m_{n+1}}) = \mathbf{X}^{\mathrm{GL}_{n+1}}_{(a_1,\dots,a_n,0)}(t_{m_1},\dots,t_{m_{n+1}})(t_{m_1}\cdots t_{m_{n+1}})^n$$

and

$$\mathbf{X}^{\mathrm{GL}_n}_{(-b_n,\dots,-b_1)}(t_{m_{n+2}},\dots,t_{m_{2n+1}}) = \mathbf{X}^{\mathrm{GL}_n}_{(b_1,\dots,b_n)}(t^{-1}_{m_{n+2}},\dots,t^{-1}_{m_{2n+1}}).$$

So the zeta integral now reads:

$$\sum_m \operatorname{sgn}(m) \sum_{a_i b_j} \left\{ \frac{1}{\prod\limits_{\substack{l<k \\ \text{not both in same} \\ \text{subset of } t\text{'s}}} (t_l - t_k)} \mathbf{X}^{\mathrm{GL}_{n+1}}_{(a_1,\dots,a_n,0)}(t_{m_1},\dots,t_{m_{n+1}}) \right.$$

$$\times \mathbf{X}^{\mathrm{GL}_n}_{(a_1,\dots,a_n)}(u_1,\dots,u_n)\mathbf{X}^{\mathrm{GL}_n}_{(b_1,\dots,b_n)}(t^{-1}_{m_{n+2}},\dots,t^{-1}_{m_{2n+1}})$$

$$\left. \times \mathbf{X}^{\mathrm{GL}_n}_{(b_1,\dots,b_n)}(v_1,\dots,v_n)(t_{m_1}\cdots t_{m_{n+1}})^n \right\}.$$

Now, it is well known ([Mac 1] (4.3), p. 33) that

$$\sum_{a_i} \mathbf{X}^{\mathrm{GL}_{n+1}}_{(a_1,\dots,a_n,0)}(t_{m_1},\dots,t_{m_{n+1}})\mathbf{X}^{\mathrm{GL}_n}_{(a_1,\dots,a_n)}(u_1,\dots,u_n) = \frac{1}{\prod\limits_{\substack{1\le i\le n+1 \\ 1\le j\le n}} (1 - t_{m_i}u_j)}$$

and

$$\sum_{b_j} \mathbf{X}^{\mathrm{GL}_n}_{(b_1,\dots,b_n)}(t^{-1}_{m_{n+2}},\dots,t^{-1}_{m_{2n+1}})\mathbf{X}^{\mathrm{GL}_n}_{(b_1,\dots,b_n)}(v_1,\dots,v_n) = \frac{1}{\prod\limits_{\substack{n+2\le i\le 2n+1 \\ 1\le j\le n}} (1 - t^{-1}_{m_i}v_j)}.$$

Let $\square_m$ denote the set $\{m_i\}_{i=1}^{n+1}$ defining the minor m ($\square_m \subset \{1,\dots,2n+1\}$). Then the local zeta integral is

$$\sum_m \operatorname{sgn}(m) \sum_{a_i b_j} \frac{1}{\prod\limits_{\substack{l<k \\ \text{not}(l \text{ and } k)\notin\square_m \\ \text{not}(l \text{ and } k)\in\square_m}} (t_l - t_k)} \frac{1}{\prod\limits_{\substack{l\in\square_m \\ 1\le j\le n}} (1 - t_l u_j)} \frac{1}{\prod\limits_{\substack{l\notin\square_m \\ 1\le j\le n}} (1 - t_l^{-1} v_j)}.$$

Clearly, the common denominator over all minors is

$$\frac{1}{\prod\limits_{\substack{l<k \\ l,k\in\{1,\dots,2n+1\}}} (t_l - t_k)} \frac{1}{\prod\limits_{\substack{1\le i\le 2n+1 \\ 1\le j\le n}} (1 - t_i^{-1} v_j)} \frac{1}{\prod\limits_{\substack{1\le i\le 2n+1 \\ 1\le j\le n}} (1 - t_i u_j)}.$$

For each minor, we have the following element of the local integral:

$$\frac{\operatorname{sgn}(m) \prod\limits_{\substack{l<k \\ l,k\in\square_m}} (t_l - t_k) \prod\limits_{\substack{l<k \\ l,k\notin\square_m}} (t_l - t_k) \prod\limits_{\substack{j \\ l\in\square_m}} (1 - t_l^{-1} v_j) \prod\limits_{\substack{j \\ l\notin\square_m}} (1 - t_l u_j) \prod\limits_{l\in\square_m} (t_l)^n}{\prod\limits_{i,j} (1 - t_i^{-1} v_j) \prod\limits_{i,j} (1 - t_i u_j) \prod\limits_{\substack{l<k \\ l,k}} (t_l - t_k)}.$$

Denote by $*\,(m)$ the numerator and let $(**)$ be $\sum_m *\,(m)$.

We shall show now that

$$(**) = \prod_{j,i} (1 - u_i v_j) \prod_{\substack{l<k \\ l,k}} (t_l - t_k).$$

For that, we regard $(**)$ as a polynomial in the u_i's.

Step 1. For any i and any j, v_j^{-1} is a root of $(**)$ as a polynomial in u_i. Thus $\prod (u_i - v_j^{-1})$ divides $(**)$ and, since the degree of u_i is n,

$$(**) = C \prod_{j,i\in\{1,\dots,n\}} (u_i - v_j^{-1}).$$

Step 2. The constant term of the polynomial $(**)$ is

$$\prod_{\substack{l<k \\ l,k\in\{1,\dots,2n+1\}}} (t_l - t_k).$$

So the polynomial $(**)$ is

$$\prod_{j,i\in\{1,\dots,n\}} (1 - u_i v_j) \prod_{\substack{l<k \\ l,k\in\{1,\dots,2n+1\}}} (t_l - t_k).$$

From this, it follows that the zeta integral equals

$$\frac{\prod_{i,j}(1-u_iv_j)}{\prod_{i,l}(1-t_lu_i)\prod_{l,j}(1-t_l^{-1}v_j)}\,.$$

Proof of Step 1. Put v_j^{-1} in place of u_i, then multiply and divide (*) by $(\prod_{l\notin\square_m}t_l^{-1})v_j^n$. Then $\prod_{l\notin\square_m}(1-t_lu_i)$ becomes $\prod_{l\notin\square_m}(1-t_l^{-1}v_j)(-1)^n$. For this j, $\prod_{l\in\square_m}(1-t_l^{-1}v_j)$ also appears in (*), and we may take

$$\prod_{l\in\{1,\dots,2n+1\}}(1-t_l^{-1}v_j)(-1)^nv_j^n$$

outside of the summation. Now it is left to show

$$\sum_m \prod_{\substack{l<k\\ l,k\in\square_m}}(t_l-t_k)\prod_{\substack{l<k\\ l,k\notin\square_m}}(t_l-t_k)\prod_{\substack{k\\ l\in\square_m\\ k\neq j}}(1-t_l^{-1}v_k)$$
$$\times\prod_{\substack{k\\ l\notin\square_m\\ k\neq i}}(1-t_lu_k)\prod_{l\in\square_m}(t_l)^n\prod_{l\notin\square_m}(t_l)=0.$$

To show that this sum is zero, we shall express the coefficients of the v's and the u's in terms of determinants in the variables t, with each determinant zero. First we need some notation.

Let $x_1,\dots,x_n$ be n indeterminants, and $\lambda=(n_1,\dots,n_n)$ a partition. Denote

$$\begin{vmatrix} x_1^{n_1} & \cdots & x_n^{n_1}\\ \vdots & & \vdots\\ x_1^{n_n} & \cdots & x_n^{n_n}\end{vmatrix}$$

by $(n_1,\dots,n_n)[x_1,\dots,x_n]$ or simply by $(n_1,\dots,n_n)$. Then $\prod_{1<k}(x_1-x_k)$ is $(n-1,\dots,0)[x_1,\dots,x_n]$, or just $(n-1,\dots,0)$. Denote the ith elementary symmetric function of the x_i's by $e_i[x_1,\dots,x_n]$ or by e_i, that is, e_i is the ith coefficient of $\prod_{j=1}^n(1-x_jt)$. Also, $e_i^{-1}[x_1,\dots,x_n]$ denotes $e_i[x_1^{-1},\dots,x_n^{-1}]$. (We agree $e_0=1$.)

LEMMA.

(1) $e_i\cdot(n-1,\dots,0)=\sum_{\epsilon\text{ in }I_i}(n-1,\dots,0)+\epsilon$,

(2) $e_i^{-1}\cdot(n-1,\dots,0)=\sum_{\epsilon\text{ in }I_i}(n-1,\dots,0)-\epsilon$,

where I_i is the set of all n tuples $(\epsilon_{n-1},\dots,\epsilon_0)$ of zeroes or ones such that $\sum\epsilon_j=i$.

PROOF. (1) is a simple consequence of the differential equations satisfied by characters of highest weight modules of GL_n [Shin], and (2) is a consequence of the fact that

$$\mathbf{X}^{GL_n}_{(a_1,\ldots,a_n)}(x_1,\ldots,x_n) = \mathbf{X}^{GL_n}_{(-a_n,\ldots,-a_1)}(x_1^{-1},\ldots,x_n^{-1}).$$

We say that the weight $(n_1,\ldots,n_n)$ "includes" $(m_1,\ldots,m_n)$ if, for all $i \in (1,\ldots,n)$, $n_i \geq m_i$. Also,

$$x_1 \cdot \overset{l}{\hat{\cdot}} \cdot x_n$$

denotes the product of the x's excluding x_l. We now go back to the proof.

The coefficient of

$$u_1^{i_1} \cdot \overset{k}{\hat{\cdot}} \cdot u_n^{i_n} \; v_1^{j_1} \cdot \overset{l}{\hat{\cdot}} \cdot v_n^{j_n}$$

where $i_1,\ldots,i_n \in \{1,\ldots,n\}$, $j_1,\ldots,j_n \in \{1,\ldots,n+1\}$ is

$$\left(\prod_{i\in\{n+2,\ldots,2n+1\}} t_{m_i}\right)\{(n-1,\ldots,0)(e_{i_1} \cdot \overset{l}{\hat{\cdot}} \cdot e_{i_n})\}[t_{m_{n+2}},\ldots,t_{m_{2n+1}}]$$

$$\cdot (t_{m_1},\ldots,t_{m_{n+1}})^n\{(n,\ldots,0)(e_{j_1}^{-1} \cdot \overset{l}{\hat{\cdot}} \cdot e_{j_n}^{-1})\}[t_{m_1},\ldots,t_{m_{n+1}}].$$

Note

$$\left(\prod_{i\in\{n+2,\ldots,2n+1\}} t_{m_i}\right)\{(n-1,\ldots,0)(e_{i_1} \cdot \overset{l}{\hat{\cdot}} \cdot e_{i_n})\}[t_{m_{n+2}},\ldots,t_{m_{2n+1}}]$$

$$= \{(n,\ldots,1)(e_{i_1} \cdot \overset{l}{\hat{\cdot}} \cdot e_{i_n})\}[t_{m_{n+2}},\ldots,t_{m_{2n+1}}]$$

and

$$(t_{m_1},\ldots,t_{m_{n+1}})^n\{(n,\ldots,0)(e_{j_1}^{-1} \cdot \overset{l}{\hat{\cdot}} \cdot e_{j_n}^{-1})\}[t_{m_1},\ldots,t_{m_{n+1}}]$$

$$= \{(2n,\ldots,n)(e_{j_1}^{-1} \cdot \overset{l}{\hat{\cdot}} \cdot e_{j_n}^{-1})\}[t_{m_1},\ldots,t_{m_{n+1}}].$$

In view of the lemma above,

$$(n,\ldots,1)(e_{i_1} \cdot \overset{l}{\hat{\cdot}} \cdot e_{i_n})$$

is a sum of weights all "included in" $(2n-1,\ldots,n)$, and

$$(2n,\ldots,n)(e_{j_1}^{-1} \cdot \overset{l}{\hat{\cdot}} \cdot e_{j_n}^{-1})$$

is a sum of weights which all "include" $(n+1,\ldots,1)$. Thus the coefficient of $u_1^{i_1}\cdots u_n^{i_n}v_1^{j_1}\cdots v_n^{j_n}$ is a sum of weights all of length $2n+1$ with entries from 1 to

$2n-1$ so at least one integer appears twice. But then look at each such weight as a minor in a determinant with two rows the same, and sum over all minors with the corresponding sign of the minor which depends only on the choice of the indeterminants. This concludes Step 1.

Proof of Step 2. The constant term of the polynomial (**) is

$$\sum \operatorname{sgn}(m) \prod_{\substack{l<k \\ l,k \in \{1,\dots,n+1\}}} (t_{m_l} - t_{m_k})$$
$$\times \prod_{\substack{l<k \\ l,k \in \{n+2,\dots,2n+1\}}} (t_{m_l} - t_{m_k})(t_{m_1},\dots,t_{m_{n+1}})^n$$

or

$$\sum_n \operatorname{sgn}(m)\{(2n,\dots,n)[t_{m_1},\dots,t_{m_{n+1}}]\}\{(n-1,\dots,0)[t_{m_{n+2}},\dots,t_{m_{2n+1}}]\}$$
$$= (2n,\dots,0)[t_1,\dots,t_{2n+1}]$$

and this concludes Step 2.

Now recall, we originally put $u_i = u_i q_{F_v}^{-s}$ and $v_j = v_j q_{F_v}^{-s}$. Thus we have shown finally that

$$\zeta(W,f^\tau,s) = \frac{\prod (1 - u_i v_j q_{F_v}^{-2s})}{\prod_{\substack{k\in\{1,\dots,2n+1\} \\ j\in\{1,\dots,n\}}} (1 - t_k^{\pi^{-1}} v_j q_{F_v}^{-s}) \prod_{\substack{k\in\{1,\dots,2n+1\} \\ i\in\{1,\dots,n\}}} (1 - t_k^{\pi} u_i q_{F_v}^{-s})}.$$

(2.4) Zeta integrals and L-functions

We may summarize the results of sections (2.2) and (2.3) as follows. Let

$$\zeta(W,f^\tau,s) = \int_{T_d^0\setminus T_d} \mathbf{W}_\chi(t)\mathbf{W}_\tau^0(t)\delta_P^{1/2}(t)|t|^{s-1/2}\delta_H^{-1}(t)\,dt.$$

Let r^0 be the representation of ${}^LG'^0 = \mathrm{GL}_{2n+1}(C) \times \mathrm{GL}_n(C) \times \mathrm{GL}_n(C)$ of type $\rho_{2n+1} \otimes \rho_n \otimes 1$ where ρ_n is the standard representation of $\mathrm{GL}_n(C)$. Consider the extension of r^0 to r on ${}^LG' = {}^LG'^0 \times \Gamma$ such that

$$r(g,a,b,\gamma) = \begin{array}{|c|c|}\hline & g^\gamma \otimes b \\ \hline g\otimes a & \\ \hline\end{array} \quad \text{and} \quad r(g,a,b,1) = \begin{array}{|c|c|}\hline g^\gamma \otimes b & \\ \hline & g\otimes a \\ \hline\end{array}.$$

Define

$$d_\tau(s) = \begin{cases} \prod_{i<j} (1 - t_i^\tau t_j^\tau q_F^{-2s})^{-1} \prod_i (1 - t_i^\tau q_F^{-s^{-1}}) & \text{non-split case,} \\ \{\prod (1 - u_i v_j q_F^{-s})\}^{-1} & \text{split case.} \end{cases}$$

THEOREM 1. *For unramified data (i.e., unramified π, τ, W, f^τ) we have*

$$d_\tau(2s)\zeta(W, f^\tau, s) = L(s, \pi \times \tau, r),$$

where $L(s, \pi \times \tau, r) = (I - \det r(t) q_F^{-s})^{-1}$, and t is the class in ${}^LG'$ corresponding to the representation $\pi \times \tau$.

We note that $d_\tau(2s)$ is the L-function for $\mathrm{Res}_F^E \mathrm{GL}_n$ which naturally arises from the adjoint action of that group on the unipotent radical of P; cf. [Sh1], p. 564. It is also the global factor which "normalizes" the Eisenstein series $E_{f^\tau}(h,s)$.

3. Intertwining Operators

To analyze our Eisenstein series, for a function in the space induced from a maximal parabolic subgroup of $H = U_{n,n}$, we have to understand the analytic behavior of one intertwining operator. Locally, we are concerned with the operator

$$f(g) \to \int_{\bar{N}_{w_0}} f(g w_0 n)\, dn$$

where $f \in \mathrm{Ind}_P^H \tau \cdot |\ \ |^s$, w_0 is

$$\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix},$$

$\bar{N}_{w_0}$ is the unipotent radical of the parabolic opposite to P, and τ is an irreducible representation of $\mathrm{Res}_F^E \mathrm{GL}_n$. We denote this operator by $A(s, \tau, w_0)$. It is well known ([Sh 2]) that the integral defining $A(s, \tau, w_0)$ converges absolutely in some right half plane $\mathrm{Re}(s) \gg 0$.

(3.1) Reduction to cuspidal induced from maximal parabolic

Our purpose in this section is to reduce the analysis of the intertwining operator $A(s, \tau, w_0)$ to the case where τ is cuspidal and P is a maximal parabolic. As mentioned already in the Introduction, we carry out the general reduction argument of [Sh 1] (Theorem 2.1.1) in order to explicitly describe the intertwining operators of cuspidal-maximal parabolic type that occur in $A(s, \tau, w_0)$. Because the split case, which concerns only GL_{2n}, is actually treated in [Sh 3], we deal here only with the non-split case, i.e., with the case H equals the quasi-split unitary group $U_{n,n}$.

First we introduce some notation. Denote by $\epsilon_1, \ldots, \epsilon_n$ the Z basis of $X(T_d)$. Let Δ denote a basis of the relative root system Σ. This root system is of type C_n. To each ψ a subset of Δ there corresponds a parabolic P_ψ of $U_{n,n}$ defined over F. For any ψ' and ψ subsets of Δ, let $W(\psi, \psi')$ denote the set of all Weyl group elements taking ψ to ψ'. For $w \in W(\psi, \psi')$ set

$$\bar{N}_w = N_\psi^- \cap w^{-1} N_{\psi'}^+ w,$$

where

$$N_\psi^{\pm} = \prod_{\alpha \in \Sigma^{\pm} - \Sigma_\psi^{\pm}} U^\alpha,$$

U^α the root subgroup of α, Σ^+ (resp. Σ^-) the positive (resp. negative) roots of Σ and Σ_ψ^+ (resp. Σ_ψ^-) the positive (resp. negative) roots of Σ in the span of ψ (resp. $-\psi$). Now fix $\Delta = \{\epsilon_i - \epsilon_j, 2\epsilon_n\}_{i \neq j}$, and let $\psi = \psi' = \{\epsilon_i - \epsilon_j\}_{i \neq j}$. Note $w_0 \in W(\psi, \psi')$.

Let $P_{(n)}$ denote $P_{\{\epsilon_i - \epsilon_j\}_{i \neq j}}$, which is a maximal parabolic subgroup whose Levi component is $\mathrm{Res}_F^E \mathrm{GL}(n)$. Also, let $\delta_{(n)}$ denote the modulus function of that parabolic. Note that $\bar{N}_{w_0}$ is the unipotent radical of the parabolic opposite to $P_{(n)}$.

Suppose now τ is an irreducible representation of $\mathrm{GL}_n(E)$. If τ is non-cuspidal, then it is contained in an induced-from-cuspidal representation of a parabolic subgroup of $\mathrm{GL}_n(E)$ of type $(n_1, \ldots, n_k)$, i.e., there exist cuspidal representations τ_i of GL_{n_i} such that τ is contained in $\mathrm{Ind}\, P_{(n_1, \ldots, n_k)}^{\mathrm{GL}_n(E)} \tau_1 \otimes \cdots \otimes \tau_k$. Note that any standard upper parabolic subgroup in $\mathrm{GL}_n(E)$ defines a standard upper parabolic subgroup of $U_{n,n}$ included in $P_{(n)}$. Let θ be the subset of Δ which corresponds to the parabolic of $U_{n,n}$ which corresponds to $(n_1, \ldots, n_k)$. Then we denote θ also by $(n_1, \ldots, n_k)$. Similarly, let θ' be $(n_k, \ldots, n_1)$. Then $w_0 \in W(\theta, \theta')$, and we may regard $A(s, \tau, w_0)f$ as the restriction to $\mathrm{Ind}\, \tau |\ |^s$ of a certain intertwining operator belonging to the parabolic P_θ. More precisely, regard f as an element of

$$\mathrm{ind}_{P_{(n_1, \ldots, n_k)}}^{U_{n,n}} \tau_1 |\ |^s \otimes \cdots \otimes \tau_k |\ |^s \delta_{(n_1, \ldots, n_k)}^{-1/2} \delta_{(n)}^{1/2};$$

then the action of our intertwining operator coincides with the action of the intertwining operator $A((s_1, \ldots, s_k), \tau_1, \ldots, \tau_k, w_0)$ on f, where $(s_1, \ldots, s_k)$ is the k tuple corresponding to $\delta_{(n_1, \ldots, n_k)}^{-1/2} \delta_{(n)}^{1/2}(s, \ldots, s)$. Note that in this section, for all induced representations the Levi component acts on the right (following [Sh 2]).

PROPOSITION 2 (The inductive step). *The intertwining operator* $A(s,\tau,w_0)$ *may be expressed as the product*

$$A(s',\tau',w')\left\{\prod_{k>j} B((s_k',s_j'),\tau_j,\bar{\tau}_k,w_{k,j})\right\}A(s_k',\tau_k,w_k).$$

Here $A(s_k',\tau_k,w_k)$ is an intertwining operator for U_{n_k,n_k} defined exactly as for $U_{n,n}$ above, except that now τ_k (in place of τ) is a cuspidal representation of $\mathrm{Res}_F^E\,\mathrm{GL}_{n_k}$ (in place of $\mathrm{Res}_F^E\,\mathrm{GL}_n$). s_k' is a translation of s_k computed below. Also, $B((s_k',s_j'),\tau_j,\bar{\tau}_k,w_{k,j})$ is the intertwining operator for $\mathrm{GL}_{n_k+n_j}(E)$ defined by

$$f\to\int_{\bar{U}^{P_{n_j,n_k}}} f(w_{k,j}n)\,dn$$

where

$$f\in \mathrm{ind}_{P_{n_j,n_k}}^{\mathrm{GL}_{n_k+n_j}(E)}(\tau_j|\ \ |^{s_k'})(\bar{\tau}_k|\ \ |^{-s_j'}),$$

P_{n_j,n_k} is the parabolic subgroup of $\mathrm{GL}_{n_k+n_j}(E)$ of type (n_j,n_k), $\bar{U}^{P_{n_j,n_k}}$ the unipotent radical of the parabolic opposite to P_{n_j,n_k}, $\bar{\tau}_k(g)=\tau_k(J_k\bar{g}^{t^{-1}}J_k)$, and

$$w_{k,j}=\begin{bmatrix}0 & I_{n_j}\\ I_{n_k} & 0\end{bmatrix}.$$

Finally, $A(s',w',\tau')$ is the intertwining operator for $U_{n-n_k,n-n_k}$ defined exactly as for $U_{n,n}$ above, except that now τ' is a representation of $P_{(n-n_k)}$ trivial on its unipotent radical and, as a representation of its Levi component $\mathrm{Res}_F^E\,\mathrm{GL}_{n-n_k}$, it is induced from a cuspidal representation $\tau_1\otimes\cdots\otimes\tau_{k-1}$ on a parabolic of type $(n_1,\ldots,n_{k-1})$.

PROOF. We follow the computation of Lemma 2.1.2 [Sh 2] to decompose $w\bar{N}_w$ into some lower-dimensional groups. At each stage we have a subset θ_i of Δ and w_i' an element of the Weyl group. We look for a simple root $\alpha\notin\theta_i$ such that $w_i'(\alpha)\in\Sigma^-$. We add this root to θ_i and denote by Ω_i the set $\theta_i\cup\{\alpha\}$. Now $w_{1,\Omega_i}w_{1,\theta_i}=w_i$ passes θ_i to its conjugate $\bar{\theta}_i$ in Ω_i (where w_{1,Ω_i} (resp. w_{1,θ_i}) is the longest element in W_{Ω_i} (resp. W_{θ_i})). Then w_{i+1}' is $w_i'w_i^{-1}$, and $\theta_{i+1}=\bar{\theta}_i$.

Here $\theta_1=(n_1,\ldots,n_k)$. Pick $\alpha=2\epsilon_n\notin\theta_1$ so $w_1'(2\epsilon_n)=-2\epsilon_n$ ($w_1'=w_1$), and $\Omega_1=\theta_1\cup\{2\epsilon_n\}$. This stage produces the first intertwining operator, which flips τ_k with its contragredient. Then $\theta_2=\theta_1$. In the next stage, pick α to be the root that sits "between" the $(k-1)$st block and the kth block. This stage produces the intertwining operator that "flips" GL_{n_k} and $\mathrm{GL}_{n_{k-1}}$ in $\mathrm{GL}_{n_{k-1}+n_k}$. Then $\theta_3=$

$(n_1, \ldots, n_k, n_{k-1})$. Next we pick α to be the root that "sits" between the $(k-2)$nd block and the $(k-1)$st block of θ_3 (which is GL_{n_k}). We go on until the block GL_{n_k} sits in first place. Then $\theta_{k+1} = (n_k, n_1, \ldots, n_{k-1})$.

Using this decomposition and Theorem 2.1.1 in [Sh 2], we can write $A(s, w_0, \tau)$ as the composition of operators of the following three types. First comes the operator

$$f \to \int_{\bar{U}^{P_{n_k}}} f(w_k n)\, dn$$

where $f \in \mathrm{ind}_{P_{(n_k)}}^{U_{n_k,n_k}} \tau_k|\quad|^{s_k} \delta_{P_{(n_k)}}^{-1/2} \delta_{P_{\theta_1}}^{1/2}$. The second intertwining operator is a "B" type operator:

$$f \to \int_{\bar{U}^{P_{n_{k-1},n_k}}} f(w_{k,k-1} n)\, dn$$

where $f \in \mathrm{ind}_{P_{n_{k-1},n_k}}^{\mathrm{GL}_{n_k+n_{k-1}}(E)} (\tau_{k-1}|\quad|^{s_{k-1}})(\bar{\tau}_k|\quad|^{-s_k}) \delta_{P_{\theta_2}}^{1/2} \delta_{P_{n_{k-1},n_k}}^{-1/2}$, and the next $k-2$ operators are of the same type. Finally, the $k+1$ intertwining operator $A(s', \tau', w')$ is

$$f \to \int_{\bar{U}^{P_{n-n_k}}} f(w' n)\, dn$$

where $f \in \mathrm{ind}_{P_{n-n_k}}^{U_{n-n_k,n-n_k}} \tau_1|\quad|^{s_1} \otimes \cdots \otimes \tau_{k-1}|\quad|^{s_{k-1}} \delta_{P_{\theta_{k+1}}}^{1/2} \delta_{P_{n-n_k}}^{-1/2}$.

This completes the proof of Proposition 2 (more details are given in [Ta]).

Concluding Remarks. (1) The intertwining operator $A(s', \tau', w')$ is of the same type as $A(s, \tau, w_0)$, except that we are dealing now with a partition of $n - n_k$ in place of n and therefore we can continue this process inductively. On the other hand, the intertwining operators of type 1 and 2 above really correspond to the case of maximal parabolics and cuspidal representations of the Levi components. Therefore, if we continue this reduction inductively in $A(s', \tau', w')$, we indeed obtain an expression for $A(s, \tau, w_0)$ in terms of operators for maximal parabolics and cuspidal representations.

(2) The argument given here depends only on the root system, and hence is also good for split groups of type C_n.

(3.2) Analytic behavior of intertwining operator for $U_{n,n}$

Our purpose in this section is to study the operator $A(s, \tau, w_0)$ in case P is maximal parabolic in $H = U_{n,n}$ and τ is a cuspidal representation of $\mathrm{Res}_F^E \mathrm{GL}_n$ on some space V_τ.

It is known (cf. [Sh 2], Theorem 2.2.1) that there exists a polynomial $p(\tau,s)$ such that $p(\tau,s)A(s,\tau,w_0)$ is holomorphic. This polynomial is described in terms of the central character of τ. Using the methods of Olshanskii [Olsh] we shall give some results with a similar polynomial but of lower degree.

Following now the notations of [Olsh], our induced representation will have the Levi component acting from the left (instead of the right, as in [Sh 2]). Therefore,

$$A(s,\tau,w_0)f(e) = \int_{U^P} f(wn)\,dn$$

where U^P is the unipotent radical of P. The integral is known to converge absolutely in some half plane $s \gg 0$.

The theorem below is stated for $n = 2$, but should be proved in general with exactly the same argument.

THEOREM 2. *Suppose $[E:F]$ is an unramified extension, τ cuspidal, and $n = 2$. Let f be in* $\operatorname{Ind}\tau|\ |^s$, *such that $f(K)$ is in $C[q_F^s, q_F^{-s}]$. Then*

(1) $(1 - q_F^{-4s}\chi(\omega))A(s,\tau,w_0)f(e)$ *is in $C[q_F^s, q_F^{-s}]$, where χ is the central character of τ, evaluated at the diagonal element ω.*

(2) *For f having compact support outside $P\bar{U}^P$, $A(s,\tau,w_0)f(e)$ is in $C[q_F^s, q_F^{-s}]$.*

REMARKS. (1) We say that the $A(s,\tau,w_0)f$ has some analytic property (*) if all complex valued functions $s \to \langle A(s,\tau,w_0)f(g), v\rangle$ for all v in V_τ have the property (*).

(2) Restrict τ to $P \cap K$ and denote by $V_\wedge$ the space $\operatorname{Ind}_{P\cap K}^K \tau$. Then for f in V_τ, $f_{|K}$ is in $V_\wedge$. Moreover, for ϕ in $V_\wedge$ define $(i_\tau\phi)(g) = \tau(p)|p|_E^s\phi(k)$ where $g = pk$. Then $i_\tau\phi$ is in V_τ and i_τ is an isomorphism of $V_\wedge \to V_\tau$.

(3) The relative root system of H, that is, the root system of H with respect to the maximal split torus, is of type C_2. Denote by W_H its Weyl group and by W_M the Weyl group of GL_2. Then $W_M\backslash W_H/W_M$ contains three double cosets, with representatives

$$w_0 = \begin{matrix} - & - & - & 1 \\ - & - & 1 & - \\ - & 1 & - & - \\ 1 & - & - & - \end{matrix}, \quad w_1 = \begin{matrix} 1 & - & - & - \\ - & - & 1 & - \\ - & 1 & - & - \\ - & - & - & 1 \end{matrix} \quad \text{and} \quad w_2 = 1.$$

To show this, we look at the action of W_H on $C[t_1, t_2, t_1^{-1}, t_2^{-1}]$. Then the double coset of w_1 takes t_1 to t_1^{-1} or t_2 to t_2^{-1}, the double coset of w_0 takes both t_1 and t_2 to their inverses, the double coset of w_2 takes t_1 to t_2, and that is all of W_H.

(4) By the first remark, we can replace $w_0 n$ by $w' w_0 n$ where w' is

$$\left(\begin{array}{c|c} J_n & \\ \hline & J_n \end{array}\right)$$

to get an integral $\int_{U^P} f(J_{2n} n)\, dn$ with the same analytic properties. Henceforth, we write w for J_{2n}.

PROOF. We follow an idea of Olshanski [Olsh]. Consider a collection of open sets $\{X_i\}$ in $P\backslash H$ covering the points $\overline{w}_i = Pw_i$, $i = 0,1,2$ (where the w_i's are the coset representatives above for $W_M \backslash W_H / W_M$). Then the set of right translations of $\{X_i\}$ by elements in P is a cover of $P\backslash H$, since $H = \bigcup_{W_M \backslash W_H / W_M} PwP$. The fact that $P\backslash H = P \cap K\backslash K$ is a compact space implies one can find a finite subcover denoted $x_1', \ldots, x_q'$ (where $X_j' = X_i p'$ for some p' in P). Define, for $i = 0,1,2$,

$$N_1^i = \overline{U}^p \cap w_i \overline{U}^p w_i, \qquad N_2^i = P \cap w_i \overline{U}^p w_i.$$

Then

$$N_1^0 = 1, \qquad N_1^1 = \begin{matrix} 1 & - & - & - \\ - & 1 & - & - \\ - & - & 1 & - \\ iy & - & - & 1 \end{matrix}, \quad \text{where } y \text{ is in } F, \quad \text{and} \quad N_1^2 = \overline{U}^P,$$

$$N_2^0 = U^P, \quad N_2^1 = \begin{matrix} 1 & - & - & - \\ x & 1 & iy & - \\ - & - & 1 & - \\ - & - & -\bar{x} & 1 \end{matrix}, \quad \text{where } y \text{ is in } F, \quad \text{and} \quad N_2^2 = 1.$$

Let ρ denote the map $H \to P\backslash H$.

LEMMA. *For each* $i = 0,1,2$,

$$\rho : N_1^i w_i N_2^i \to P\backslash H \textit{ is a homeomorphism.}$$

PROOF. Let $U^P(F)$ (resp. $U^P(E)$) denote the F (resp. E) rational points of U^P. Take now an open set V_F in $U^P(F)$. It is the intersection of V_E open in $U^P(E)$ with $U^P(F)$. By [Olsh], $\rho_E : U(E) \to P(E)\backslash H(E)$ is a homeomorphism (the case $i = 2$ for $\mathrm{GL}_4(E)$). Now for $i = 2$, the lemma follows from the fact that

$P(E)V_E \cap H(F) = P(F)V_F$ (this is easily verified by a simple computation) and, as for the other cases, note that $N_1^i w_i N_2^i = \bar{U}^P w_i$.

Returning to the proof of the theorem, pick open compact neighborhoods U_1^i (resp. U_2^i) of the identity in N_1^i (resp. N_2^i). By the lemma, the image of $U_1^i w_i U_2^i$ is open in $P\backslash H$, and it covers $\bar{w}_i = Pw_i$. Recall X_i', $i = 1$ to q is the finite cover on $P\backslash H$ described above. So $K = \bigcup_{i=1}^{q} \{P \cap K\} X_i'$, and we can write ϕ in V_λ as a sum of ϕ_i's with ρ (support (ϕ_i)) $\subset X_i'$. Without loss of generality, we suppose $\phi = \phi_i$ so support of $i_\tau \phi$ is in $PU_1^i w_i U_2^i p'$.

Define ψ_τ on $N_1^i \times N_2^i$ by $\psi_\tau(n_1, n_2) = (i_\tau \phi)(n_1 w n_2 p')$. Then it is easy to check that

(a) ψ_τ has compact support on $U_1^i \times U_2^i$. (If $n_1^i \in N_1^i$ but not in U_1^i or $n_2^i \in N_2^i$ but not in U_2^i, then $n_1^i w_i n_2^i p' \notin PU_1^i w_i U_2^i p'$ since ρ is 1 to 1.)

(b) ψ_τ is locally constant uniformly with respect to s, that is, for any u_1, u_2, there exists compact neighborhoods U_1, U_2 such that $u_1 \in U_1$, $u_2 \in U_2$ and $\psi_\tau(U_1, U_2) = c(s)$ where $c(s)$ is some constant depending on s. (This follows from the fact that ϕ is locally constant and $|\det(U_1^i w_i U_2^i p')| = |\det(p')|$.)

We continue now with the explicit computation of the intertwining integral. For each case $i = 0,1,2$ we specify a decomposition of wn as DX_i' where D is in P. So D goes out of the integral and we are left with integration of ψ_τ on X_i'.

Case 1: $i = 0$, $\phi = \phi_0$

In this case, ψ_τ has support in $wU^P p'$ where $p' = m'n'$. By a change of variables,

$$\int_{U^P} \langle f(wn), v \rangle \, dn = \int_{U^P} \langle i_\tau \phi w(m'^{-1} n m') n', v \rangle \delta_{U^P}(m') \, dn.$$

Using the identity $(wm'^{-1}w)wnp' = (wm'^{-1}w)\ wnm'n' = wm'^{-1}nm'n'$, and moving $(wm'^{-1}w)$ to the right side of the inner product $\langle\ ,\ \rangle$, the last integral equals

$$\int_{U^P} \langle i_\tau \phi(wnp'), (wm'^{-1}w^{-1})v \rangle \delta_{U^P}(m') \, dn$$

$$= \int_{U^P} \langle \psi_\tau(n), (wm'^{-1}w^{-1})v \rangle \delta_{U^P}(m') \, dn.$$

Since ψ_τ has compact support in U^P and it is locally constant with respect to s, the last integral is in $C[q_F^{-s}, q_F^{s}]$.

Case 2: $i = 1$, $\phi = \phi_1$

Use the decomposition:

$$\underbrace{\begin{matrix} 1 & \bar{y}_1 & - & - \\ - & 1 & - & - \\ - & - & 1 & -y_1 \\ - & - & - & 1 \end{matrix}}_{n_{y_1}} \quad \underbrace{\begin{matrix} -a^{-1} & - & - & 1 \\ - & 1 & - & - \\ - & - & 1 & - \\ - & - & - & a \end{matrix}}_{\rho_{n^{-1}}} \quad \underbrace{\begin{matrix} 1 & - & - & - \\ - & 1 & - & - \\ - & - & 1 & - \\ a^{-1} & - & - & 1 \end{matrix}}_{g_{a^{-1}}} \quad \begin{matrix} 1 & - & - & - \\ - & - & 1 & - \\ - & 1 & - & - \\ - & - & - & 1 \end{matrix} \quad \underbrace{\begin{matrix} 1 & - & - & - \\ y_1 & 1 & y_2 & - \\ - & - & 1 & - \\ - & - & -\bar{y}_1 & 1 \end{matrix}}_{g_{y_1 y_2}}$$

$$= \begin{matrix} - & - & - & 1 \\ - & - & 1 & - \\ - & 1 & - & - \\ 1 & - & - & - \end{matrix} \quad \begin{matrix} 1 & - & -a\bar{y}_1 & a \\ - & 1 & \times & y_1 a \\ - & - & 1 & - \\ - & - & - & 1 \end{matrix}$$

where a is in iF^*, y_2 is in iF, and $x = y_2 + y_1 a \bar{y}_1$. So, normalizing the measure dn as before, we have

$$\int_{U^P} \langle i_\tau \phi(wn), v \rangle \, dn$$

$$= \int_{iF_v^* \times E \times iF} \langle i_\tau \phi(\rho_{a^{-1}} g_{a^{-1}} w_1 g_{y_1 y_2} p'), (wm'^{-1} wn_{y_1})^{-1} v' \rangle$$

$$\times \; |a|_E^1 \delta_{U^P}(m') \, da \, dy_1 \, dy_2$$

$$= \int \langle \tau(\rho_{a^{-1}}) i_\tau \phi(g_{a^{-1}} w_1 g_{y_1 y_2} p'), (wm'^{-1} wn_{y_1})^{-1} v' \rangle$$

$$\times \; |a|_E^{-s-1/2} |a|_E^1 \delta_{U^P}(m') \, da \, dy_1 \, dy_2$$

$$\underset{\substack{da \to da^* \\ a \to a^{-1}}}{=} \int_{iF_v^* \times E \times iF} \langle \tau(\rho_a) \psi_\tau(g_a, g_{y_1 y_2}) v, (wm'^{-1} wn_{y_1})^{-1} v' \rangle$$

$$\times \; |a|_E^{s'} \delta_{U^P}(m') \, da^* \, dy_1 \, dy_2.$$

Note ψ_τ and $y_1 \to n_{y_1} v'$ are locally constant uniformly with respect to s and ψ_τ is compactly supported on $(g_a, g_{y_1 y_2})$. Therefore, the last integral is a sum of integrals

$$\int_{R \cap iF^*} \left\langle \tau \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v, v' \right\rangle |a|_E^{s'} \, d^* a$$

where R is compact and open in E (might contain 0), $R \cap iF^*$ is closed in E^* and the matrix coefficient has compact support in E^*, since τ is supercuspidal. Therefore, the last integral is in $C[q_F^{-s}, q_F^{s}]$ (and hence we have already proved part 2 of the theorem).

Case 3: $i = 2$, $\phi = \phi_2$, $w = 1$
Use the decomposition

$$\underbrace{\begin{pmatrix} -Ja^{-1} & \begin{smallmatrix} - & 1 \\ 1 & - \end{smallmatrix} \\ \begin{smallmatrix} - & - \\ - & - \end{smallmatrix} & Ja \end{pmatrix}}_{\rho_{a^{-1}}} \underbrace{\begin{pmatrix} \begin{smallmatrix} 1 & - \\ - & 1 \end{smallmatrix} & \begin{smallmatrix} - & - \\ - & - \end{smallmatrix} \\ a^{-1} & \begin{smallmatrix} 1 & - \\ - & 1 \end{smallmatrix} \end{pmatrix}}_{g_{a^{-1}}} = \begin{pmatrix} - & - & - & 1 \\ - & - & 1 & - \\ - & 1 & - & - \\ 1 & - & - & - \end{pmatrix} \begin{pmatrix} \begin{smallmatrix} 1 & - \\ - & 1 \end{smallmatrix} & a \\ \begin{smallmatrix} - & - \\ - & - \end{smallmatrix} & \begin{smallmatrix} 1 & - \\ - & 1 \end{smallmatrix} \end{pmatrix}$$

where $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $a \in N = \{a \in M_{2\times 2}(E);\ J\bar{a}^t J = -a\}$. Then

$$\begin{aligned} \int_{U_P} \langle i_\pi \phi(wn), v\rangle\, dn &= \int_{N \cap \mathrm{GL}_2(E)} \langle i_\pi \phi(\rho_{a^{-1}} g_{a^{-1}}, wm'wv\rangle\, dn \\ &= \int_{N \cap \mathrm{GL}_2(E)} \langle \tau(a^{-1}) \psi_\pi(g_{a^{-1}}), v\rangle |a|_E^{-s-1/2}\, dn \\ &= \int_{a^{-1} \in R \cap N \cap \mathrm{GL}_2(E)} \langle \tau(a^{-1}) v, v'\rangle |a|_E^{-s-1/2}\, dn \end{aligned}$$

where R is compact and open in $M_{2\times 2}(E)$. Suppose $R = M_{2\times 2}(P_F^m O_{P_E})$; then it is well known that $\mathrm{GL}_2(E) \cap R = \coprod_{j \ge 0} \mathbf{P}_F^j \phi$ where $\phi = \coprod_{k \ge 0} K_E \epsilon_{m,m+k} K_E$,

$$\epsilon_{a,b} = \begin{matrix} \omega^a & 0 \\ 0 & \omega^b \end{matrix},$$

and $\mathbf{P}_F^j = \epsilon_{j,j}$ (cf. [Olsh], p. 238). Also, $\{\coprod_{j\ge 0} \mathbf{P}_F^j \phi\} \cap N = \coprod_{j \ge 0} \mathbf{P}_F^j \{\phi \cap N\}$. Now a^{-1} in $\coprod_{j\ge 0} \mathbf{P}_F^j\{\phi \cap N\}$ means a in $\coprod_{j \ge 0} \mathbf{P}_F^{-j}\{\phi \cap N\}^{-1}$. So the last integral reads:

$$\sum_j \chi(\mathbf{P}_F^j) |\mathbf{P}_F^{-j}|_E^{-s-1/2} \int_{\mathbf{P}_F^{-j}\{\phi \cap N\}^{-1}} \langle \tau(a^{-1}) v, v'\rangle |a|_E^{-s-1/2}\, dn.$$

Using the notations of Case 2, we can write $dn = da\, dy_1\, dy_2$, so it is easy to check that $d\mathbf{P}_F^{-j} n = |\mathbf{P}_F^{-j}|_F\, dn$. Then the last sum equals

$$\sum_j \chi(\mathbf{P}_F^j) |\mathbf{P}_F^{-j}|_E^{-s-1/2} |\mathbf{P}_F^{-j}|_E^{1/2} \int_{\{\phi \cap N\}^{-1}} \langle \tau(a^{-1}) v, v'\rangle |a|_E^{-s-1/2}\, dn,$$

which is

$$\frac{1}{1 - q_F^{-(4s)}\chi(\mathbf{P}_F)} \int_{\{\phi \cap N\}^{-1}} \langle \tau(a^{-1})v, v' \rangle |a|_E^{-s-1/2}\, dn.$$

By cuspidality, $\langle \tau(a)v, v' \rangle$ has support in ZC, Z the center of GL_2 and C compact in GL_2. Thus $a^{-1} \in ZC \cap \phi \cap N$. But $ZC \cap \phi$ is compact in GL_2 (cf. [Olsh], p. 238), and N is closed in $M_{2\times 2}(E)$. So a^{-1} goes over a compact set of GL_2 and therefore the integral is in $C[q^s, q^{-s}]$. This concludes the proof of the theorem.

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